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DISCUSSION OF FLEXURE OF DOUBLE CANTILEVER BEAMS PROCEEDINGS-SEPARATE NO. 158

LU-SHIEN HU,[5] J.M. ASCE.—In the analysis of double cantilever beams, the method of moment distribution can readily be applied. The final end moments, both bending and torsional, can be obtained in two steps. The first step is to supply a temporary support to point C, the intersection point of the two cantilevers (Fig. 1). The reaction at C is then computed from the end moments obtained by the application of the method of moment distribution. Secondly, the reaction at C is removed and a specific displacement is introduced. The difference between the application of the method of moment distribution to this particular spatial frame from that of its application to ordinary plane frames is the separate recording of both bending and torsional moments at the ends of each member.

To apply the method of moment distribution the following must be known—fixed-end moments, distribution factors, and carry-over factors. In double cantilever beams, the fixed-end moments and the carry-over factors for bending are the same as for a straight beam. The carry-over factors for torsion are unity. The distribution factors for perpendicular double cantilever beams have been used in various problems.[6,7]

The distribution factors for double cantilever beams intersecting at an angle θ can be derived by considering continuity and equilibrium at point C. The result is as follows:

$$M_{CA} = \frac{1}{D} \quad K_{CA}\,(K'_{CA} + K_{CB}\sin^2\theta + K'_{CB}\cos^2\theta) \times M_1 - \frac{1}{D} K_{CA}\,(K_{CB} - K'_{CB})\sin\theta\cos\theta\, M_2 \quad (19a)$$

$$M'_{CA} = \frac{1}{D} K'_{CA}\,(K_{CB} - K'_{CB})\sin\theta\cos\theta\, M_1 + \frac{1}{D} K'_{CA} \times (K_{CA} + K_{CB}\cos^2\theta + K'_{CB}\sin^2\theta)\, M_2 \quad (19b)$$

$$M_{CB} = -\frac{1}{D} K_{CB}\,(K'_{CA} + K'_{CB})\cos\theta\, M_1 - \frac{1}{D} K_{CB} \times (K_{CA} + K'_{CB})\sin\theta\, M_2 \quad (20a)$$

$$M'_{CB} = \frac{1}{D} K'_{CB}\,(K_{CB} + K'_{CA})\sin\theta\, M_1 - \frac{1}{D} K'_{CB} \times (K_{CA} + K_{CB})\cos\theta\, M_2 \quad (20b)$$

in which

$$D = K_{CA}\,K'_{CA} + K_{CB}\,K'_{CB} + (K_{CA}\,K_{CB} + K'_{CA}\,K'_{CB}) \times \sin^2\theta + (K_{CA}\,K'_{CB} + K_{CB}\,K'_{CA})\cos^2\theta \quad (21)$$

in which M represents the bending end moment, and M' represents the torsional end moment. These moments are positive if acting as shown in Fig. 4, in

[5] Designer, Howard, Needles, Tammen & Bergendoff, New York, N. Y.

[6] "Reinforced Concrete Structures," by Dean Peabody, John Wiley & Sons, Inc., New York, N. Y., 1946.

[7] "Deflections in Gridworks and Slabs," by Walter W. Ewell, Shigeo Okubo, and Joel I. Abrams, *Transactions*, ASCE, Vol. 117, 1952, p. 869.

which they are represented by vectors of right-hand rule. In Eqs. 19, 20, and 21 flexural rigidity is represented by K, and K' denotes torsional rigidity—that is, the moment required to produce a unit rotation. For prismatic straight members with the far end fixed, $K = \frac{4\,E\,I}{L}$ and $K' = \frac{G\,J}{L}$. The expressions for J of various sections can be found elsewhere.[6,8,9,10]

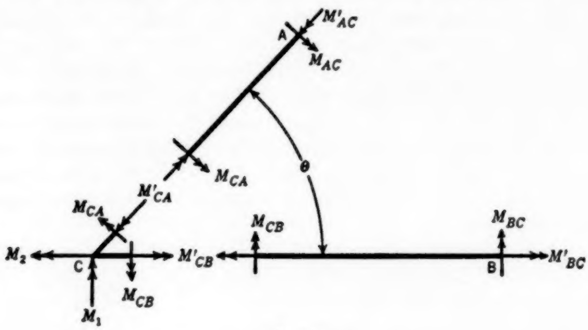

FIG. 4

Special cases can be derived from Eqs. 19, 20, and 21. When $\theta = 180°$, $\sin\theta = 0$, and $\cos\theta = 1$, the distribution factors for continuous beams are

$$M_{CA} = \frac{K_{CA}}{K_{CA} + K_{CB}} M_1 \quad (22a)$$

$$M'_{CA} = \frac{K'_{CA}}{K'_{CA} + K'_{CB}} M_2 \quad (22b)$$

$$M_{CB} = \frac{K_{CB}}{K_{CA} + K_{CB}} M_1 \quad (23a)$$

$$M'_{CB} = \frac{K'_{CB}}{K'_{CA} + K'_{CB}} M_2 \quad (23b)$$

When $\theta = 90°$, $\sin\theta = 1$, and $\cos\theta = 0$, the distribution factors for perpendicular double cantilever beams are

$$M_{CA} = \frac{K_{CA}}{K_{CA} + K'_{CB}} M_1 \quad (24a)$$

$$M'_{CA} = \frac{K'_{CA}}{K_{CB} + K'_{CA}} M_2 \quad (24b)$$

$$M_{CB} = -\frac{K_{CB}}{K_{CB} + K'_{CA}} M_2 \quad (25a)$$

$$M'_{CB} = \frac{K'_{CB}}{K_{CA} + K'_{CB}} M_1 \quad (25b)$$

[8] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., New York, N. Y., 1947.
[9] "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 857.
[10] "Torsion of Plate Girders," by F. K. Chang and Bruce G. Johnston, *ibid.*, Vol. 118, 1953, p. 337.

The method of moment distribution can be applied to double cantilever beams with end conditions other than fixed, provided that the flexural rigidity K is derived for the actual end conditions.

In the paper the sign of the term M_{0zz} in all formulas in Table 1 seems inconsistent with the sign conventions adopted by the author.

LEWIS SCHNEIDER,[11] J.M. ASCE.—The problem of double cantilever beams and similar structural configurations is frequently encountered in piping stress analysis. The method of moment distribution is a convenient device in the handling of this type of problem. This method is convenient because it can be adapted to (a) any type of loading, (b) members having different or varying cross sections, or (c) structures which are continuous. An additional advantage to the method of moment distribution is that slide-rule accuracy is more than sufficient.

The basic concepts of the method of moment distribution are familiar to most engineers. Following the determination of the torsional stiffness factors, the distribution factors for unbalanced moments acting at a joint can be obtained. By determining the fixed-end moments, distributing the unbalanced moments, and applying a correction for the vertical displacements of unsupported joints (analogous to the sidesway correction for bents), it is possible to obtain the end reactions for a structure having any degree of indeterminacy.

By applying arbitrary couples at the points of support and determining the resulting angular deflection at the joint under consideration (by the application of the slope-deflection equations), it is possible to determine both the stiffness and carry-over moments for the complete structure. For example, if in the system OCB (Fig. 1(*a*)), an arbitrary moment M_{Ozy} is applied at O and moment distribution is performed, the resulting moments M_{Ozz}, M_{Bzy}, and M_{Bzz}, can be determined. This establishes a relationship that shows what magnitude of moments are produced at O and B by the application of a unit moment at O in the *yz*-plane. By writing the slope-deflection equation for the member OC, a solution for the rotation of O, caused by the application of the unit moment, is obtained. By direct proportion, the moment corresponding to a rotation of one radian can be obtained and this moment is, by definition, the stiffness factor S_{OB}. Similarly, the torsional stiffness and carry-over moments for a twisting moment at O are established. These factors can be used to acquire a solution when full fixity is not obtainable at the points of support. An illustrative problem is shown in Figs. 5 and 6, which show the sequence of operations.

In the second term of the right-hand side of Eq. 1, the term J, generally used to represent the polar moment of inertia of a section, is used to define the ratio of proportionality between the torsional moment and the angular deflection of a member. This definition is correct for circular shafts only. Another symbol would have been a more proper choice. If this symbol were T,

[11] Design Engr., Arthur G. McKee, Co., Union, N. J.

for a rectangular member, then

$$T = \frac{b^3\, d^3}{3.58\ (b^2 + d^2)} \qquad \text{(26)}$$

in which d would represent the greatest dimension of the rectangle.[12] For rolled steel sections, values of T have been tabulated as "K-values."[13]

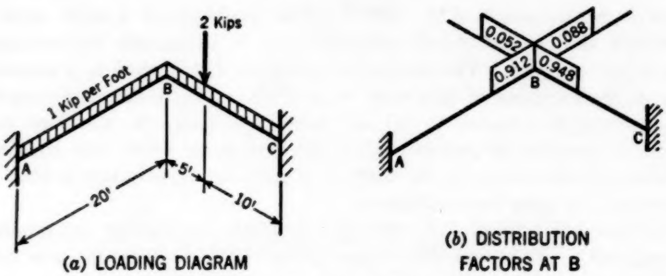

FIG. 5.—DOUBLE CANTILEVER CONCRETE BEAM

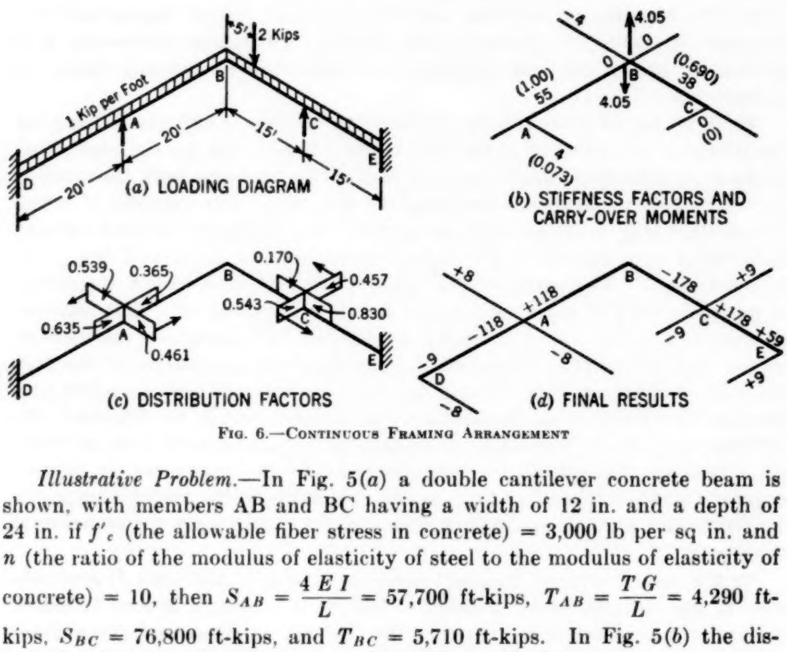

FIG. 6.—CONTINUOUS FRAMING ARRANGEMENT

Illustrative Problem.—In Fig. 5(*a*) a double cantilever concrete beam is shown, with members AB and BC having a width of 12 in. and a depth of 24 in. if f'_c (the allowable fiber stress in concrete) = 3,000 lb per sq in. and n (the ratio of the modulus of elasticity of steel to the modulus of elasticity of concrete) = 10, then $S_{AB} = \frac{4\,E\,I}{L} = 57{,}700$ ft-kips, $T_{AB} = \frac{T\,G}{L} = 4{,}290$ ft-kips, $S_{BC} = 76{,}800$ ft-kips, and $T_{BC} = 5{,}710$ ft-kips. In Fig. 5(*b*) the distribution factors at B are represented in their appropriate planes.

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[12] "Design of Reinforced Concrete in Torsion" by Paul Andersen, *Transactions*, ASCE, Vol. 103, 1938, p. 1503.

[13] "Torsional Stresses in Structural Beams," Bethlehem Steel Co., Bethlehem, Pa., 1950, pp. 5–8.

To determine the vertical deflection correction, point B is deflected 1 in. and the unbalanced fixed-end moments are distributed. By the use of statics it is found that a force of 35.9 kips will produce this 1-in. deflection. The fixed-end moments resulting from the applied loads are then distributed. It is found, by the use of the principles of statics, that a force of 14.5 kips is required at B to prevent that joint from deflecting downward. By allowing a deflection $\frac{14.5}{35.9} = 0.405$ in., and superimposing the moments resulting from such a deflection, the reactions and loads will be in a condition of static equilibrium.

In Fig. 6(*a*) the member ABC is shown as a component of a continuous framing arrangement. An arbitrary couple of 100 ft-kips is applied at A and results in the final moments shown in Fig. 6(*b*). In computing the correction for the displacement of B, the deflection was found to be 0.113 in. Since

$$M_{AB} = \frac{4\,E\,I\,\theta_A}{L} + \frac{2\,E\,I\,\theta_B}{L} - \frac{6\,E\,I\,\Delta}{L^2} \qquad (27)$$

it follows that

$$\theta_B = \frac{M'_{CB}}{T_{BC}} = \frac{0}{5{,}710} = 0 \qquad (28)$$

The symbol M' represents the torsional moment on the member, $\theta_A = 0.00166$ radian and $S_{AC} = \frac{55}{0.00166} = 33{,}100$ ft-kips. The numbers in parentheses in Fig. 6(*b*) are the carry-over moments resulting from the application of a unit couple at A.

In a similar manner $T_{AC} = 3{,}690$ ft-kips, $S_{CA} = 16{,}000$ ft-kips, and $T_{CA} = 4{,}850$ ft-kips. In Fig. 6(*c*) the distribution factors are represented in their appropriate planes. Fig. 6(*d*) shows the final results of the problem.

RAY W. CLOUGH,[14] J. M. ASCE.—A subject that has received increasing attention—the analysis of structures in which the loads act normal to the plane of the structure—is treated by the author. The principal difference between the structural action in this case, in contrast to the usual condition of loads acting in the plane of the structure, lies in the torsional moments developed. For this type of loading, the moment-distribution method extended to include the effects of torsion will often provide the simplest means of analysis.[15] This method is particularly effective when applied to the type of structure considered in this paper. Computation of the moments and torques developed by the loads—including the effects of rotation of C—is straightforward (see Fig. 1(a)). Only one distribution in each plane is required to balance the moments at C, because of the fixed ends. The final step in the procedure used in this analysis is the release of a vertical holding force at joint C and the distribution of the moments developed by the resulting displacement. This computation is most readily made by computing the moments associated with

[14] Asst. Prof. of Civ. Eng., Univ. of California, Berkeley, Calif.

[15] "The Design of Reinforced Concrete Structure," by Dean Peabody, John Wiley & Sons, Inc., New York, N. Y., 1946, p. 419.

an arbitrary magnitude of the holding force. The moments that would be developed by releasing the actual holding force are determined by proportion.

As an example of the procedure, the double cantilever structure shown in Fig. 7 (*a*) is analyzed in Table 3. The first step in the analysis is the determina-

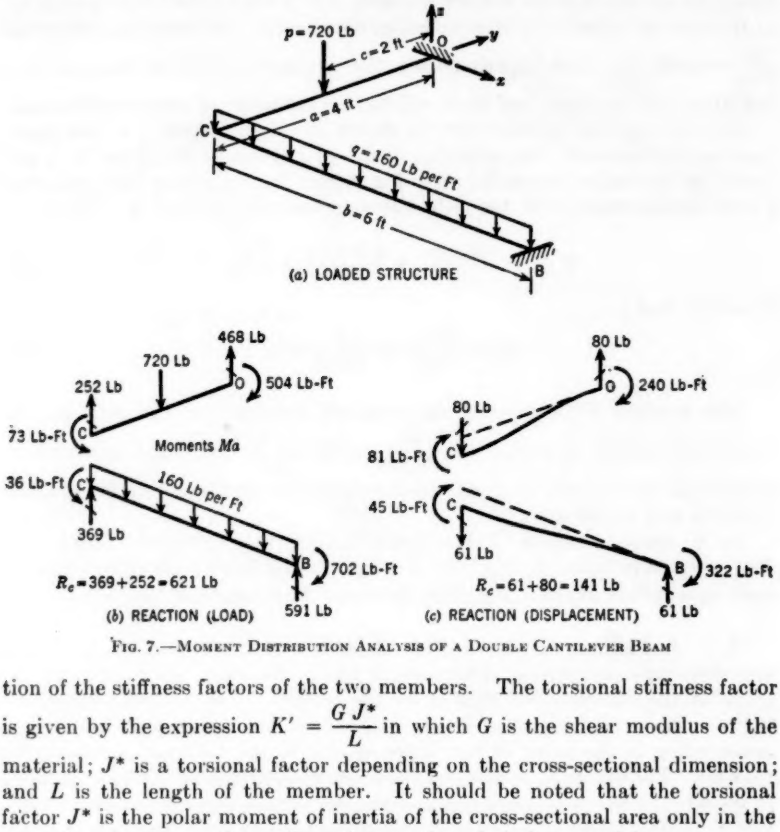


FIG. 7.—MOMENT DISTRIBUTION ANALYSIS OF A DOUBLE CANTILEVER BEAM

tion of the stiffness factors of the two members. The torsional stiffness factor is given by the expression $K' = \frac{G\,J^*}{L}$ in which G is the shear modulus of the material; J^* is a torsional factor depending on the cross-sectional dimension; and L is the length of the member. It should be noted that the torsional factor J^* is the polar moment of inertia of the cross-sectional area only in the case of a circular section. For other sections it is less—and in many cases much less[16]—than the polar moment of inertia. For rectangular sections the torsional factor is given by

$$J^* = \frac{d\,b^3}{3}\left[1 - 0.63\,\frac{b}{d}\left(1 - \frac{b^4}{12\,d^4}\right)\right] \qquad (29)$$

in which b is the width of the section, and d the depth.

In the moment-distribution solution, moments in the zx-plane and the zy-plane are considered separately. The distribution factors at joint C are

[16] "Formulas for Stress and Strain," by R. J. Roark, McGraw-Hill Book Co., Inc., New York, N. Y., 1943, p. 169.

TABLE 3.—Moment Distribution for the Loaded Structure of Fig. 7

Procedure	Moments in the xy-Plane[a]				Moments in the xz-Plane[a]			
Member	CO		BC		CO		BC	
Point	O	C	C	B	O	C	C	B
Stiffness factor (multiply by 10^6)		27.0	6.86			4.94	60.7	
Distribution factor		0.797	0.203			0.075	0.925	
Moments[a] resisting load:								
Fixed-end moment	+360	−360	0	0	0	0	−480	+480
Distribution	0	+287	+73	0	0	+36	+444	0
Carry-over	+144	0	0	−73	−36	0	0	+222
Moment, M_a	+504	−73	+73	−73	−36	+36	−36	+702
Moments[a] resisting displacement:								
Fixed-end moment	+400	+400	0	0	0	0	+600	+600
Distribution	0	−319	−81	0	0	−45	−555	0
Carry-over	−160	0	0	+81	+45	0	0	−278
Moment, M'_b	+240	+81	−81	+81	+45	−45	+45	+322
$M_b = (621/141)\, M'_b$	+1,059	+357	−357	+357	+198	−198	+198	+1,420
Total Moment $= M_a + M_b$	+1,563	+284	−284	+284	+162	−162	+162	+2,122

[a] All values are in pound-feet.

thus obtained by considering stiffnesses of the members about the two axes separately. Fixed-end moments for the two beams are computed in the usual fashion. No fixed-end torques are developed by the loads. The moment sign convention adopted for this example is as follows: When looking in the direction from C to O, or from B to C, clockwise moments and torques acting on the members are positive, and clockwise moments and torques acting on the joints are negative. Distribution of unbalanced moments at the joint and carry-over of bending moments to the supports follow the usual procedure. The carry-over factor for torques is -1, however, in contrast to the factor of $\frac{1}{2}$ used with bending moments.

After the moments at joint C have been balanced, the reactive force preventing vertical displacement of this joint may be obtained by adding the shearing forces acting at C in the two beams, as shown in Fig. 7(*b*). For the determination of the moments that will be developed in the beam when this reaction is removed, joint C is displaced through an arbitrary vertical distance Δ_c without rotation. Because of this displacement, fixed-end moments will be developed in the beams, equal to $\frac{6\,E\,I}{L^2}\,\Delta_c$. In this example, the deflection Δ_c was chosen to develop moments in member CO of 400 lb-ft. The corresponding moments in member BC are given by the relationship $\frac{M_{C\,B}}{M_{C\,O}} = \frac{(I/L^2)_{C\,B}}{(I/L^2)_{C\,O}}$ because the same deflection applies to each beam.

The reactive force at C associated with the resulting end moments (M'_b) is determined by summing shears in the beams at joint C as shown in Fig. 7(*c*). The moments and torques (M_b) that would be developed in the beams by re-

moving the actual holding force may then be computed by proportion (Table 3). The final values of end moments and torques in the beams are obtained by adding the moments produced by the loads (M_a) to the moments resulting from the displacements (M_b).

As this example demonstrates, the stresses in double cantilever beams can be determined quickly by moment distribution. In general, the advantages making the method of moment distribution popular for the analysis of structures loaded in their own planes (principally avoiding the solution of simultaneous equations) are equally beneficial for cases in which the loads are applied normal to the plane of the structure.

Contrary to the author's statement, the structure shown in Fig. 2(*g*) is stable. The reactions R and R_B are equivalent to a vertical force and a couple. This couple has components in both the *zx*-plane and the *zy*-plane. The three given reactions—M_{By}, R, and R_B can thus provide three independent reaction components (moments in the *zx*-plane and in the *zy*-plane, and a vertical force) sufficient for equilibrium in the given condition of loading.

The double cantilever structure is actually indeterminate to the sixth degree, but for the special type of structure (composed of members of which a principal axis lies in the plane of the structure) and the special case of loading (normal to the plane of the structure) considered by the author, three of the redundant reactions vanish, and the structure may be treated as though it were indeterminate to the third degree.

William A. Conwell,[17] M. ASCE.—An intriguing problem is treated by the author in this paper. The type of construction described is becoming more prevalent (1953) and has a wide field of application. Although the double cantilever beam is more complicated to analyze than the single cantilever beam, it is simple enough to be a valuable step in engineering thinking toward the solution of more complicated space structures.

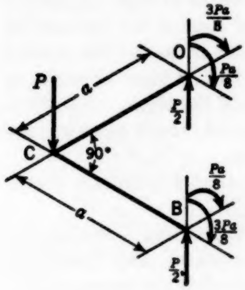
Under the heading, "General Case of Continuity," the author indicates that, in general, three reactions occur at each point of support for the vertical loads considered in the paper. A further observation might be made that, for loads inclined to the vertical, three additional reactions would occur at each of the two supports, making a total of twelve for the structure. Inspection shows, however, that any inclined load may be resolved into two components—one in the plane of the structure and the other perpendicular to it. The structure may then be analyzed separately for each of the components and the results superimposed for the final solution of the problem.

The writer prefers to think of moments acting in a plane, about or along an axis rather than "rotating," as in the author's terminology.

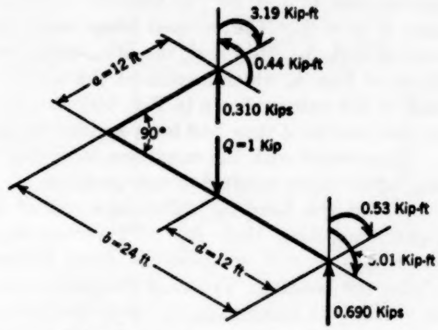
In preliminary preparation of this discussion the writer independently checked the values given in Table 2 for $A = 1$ and $A = 10$. Although the writer's results agreed in magnitude with the moments shown, the conclusion was reached that all signs for M_{Ozx} should be plus instead of minus. It is believed that one can arrive at the proper direction of the moments from several independent considerations of the structural action involved. There should

[17] Gen. Engr., Structural Eng. and Design Dept., Duquesne Light Co., Pittsburgh, Pa.

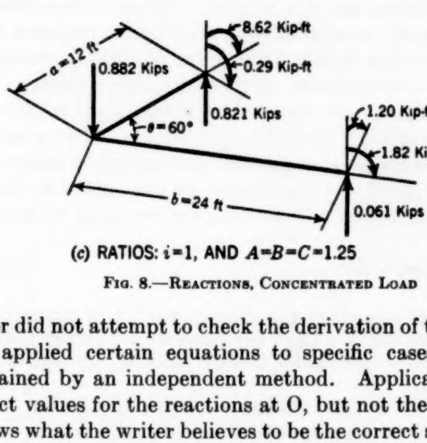
be, however, no more convincing proof than the laws of statics. Referring to Fig. 1, if one considers the case of the load P at point C of a structure for which $A = 1$ and assumes the author's value of $M_{Osy} = 3\ P\ a/8$ to be correct, application of symmetry and summation of moments about the axis of member OC yields a clockwise direction, and hence, a positive sign, for M_{Osz} (see Fig. 8(*a*)). A similar observation for the uniform load p requires that the signs for M_{Osz} be positive.



(*a*) RATIOS: $i = 1$, AND $A = B = 1$



(*b*) RATIOS: $i = 1$, AND $A = B = 1.25$



(*c*) RATIOS: $i = 1$, AND $A = B = C = 1.25$

FIG. 8.—REACTIONS, CONCENTRATED LOAD

The writer did not attempt to check the derivation of the author's equations, but instead applied certain equations to specific cases for which solutions could be obtained by an independent method. Applications of Eqs. 3 and 4 yielded correct values for the reactions at O, but not the correct sign for M_{Osz}. Fig. 8(*b*) shows what the writer believes to be the correct solution to a particular problem. Eqs. 4, applied to this problem, yield the value $M_{Osz} = +0.44$ kip-ft, a direction opposite to that shown in Fig. 8(*b*). It is evident that this reversed value of M_{Osz} would affect the value of M_{Bsz}—making it 4.13 kip-ft instead of 5.01 kip-ft—revealing a considerable error in a major moment. The writer has reluctantly reached the conclusion that, if the definition of positive moment, as illustrated in Fig. 1(*a*), is to be retained, the signs of the terms containing M_{Osz} in Eqs. 3 and 4 should be reversed. Because the left-

hand terms of Eqs. 3 to 12, inclusive, are identical (Eq. 9*c*, which has a different sign for the $M_{O_{zy}}$-term, is probably in error), it is believed that this conclusion applies to all the aforementioned equations and should be extended to Eqs. 13 and 14, whose terms are somewhat different.

With the thought of confirming Eqs. 16, an independent analysis was made of a problem to which they would apply. The result of the analysis is shown in Fig. 8(*c*). Before making the substitutions in Eqs. 16 it was decided to test them. The test applied was the reduction of the terms of Eqs. 16, for the special case of $\theta = 90°$, to identity with the terms of Eqs. 3, for the special case of $\gamma = 1$. The reduced terms were identical with those for the special case of Eqs. 3. However, the $M_{O_{zz}}$-terms from Eqs. 16 had signs opposite to those of Eqs. 3, which confirmed the writer's opinion of the signs. Substitution of the values shown in Fig. 8(*c*) showed that Eqs. 16 would have yielded correct results if they had been applied to this problem.

Experience with the equations indicates that they need a thorough checking before being applied to any problem.

Under the heading, "Development of General Equations," the author rightly explains that for "*** certain loading conditions and boundary conditions formal solutions of Eqs. 2 become somewhat unmanageable." Under the heading, "Practical Considerations," he emphasizes, again correctly, the futility of attempting to solve simultaneous equations of this type with a small slide rule. Under the heading, "Applications to Continuous Structures," he suggests an approximation for solution of the most usual case which occurs when the double cantilevers are not isolated. These observations are those which are almost invariably associated with the classical methods of analysis—the only ones available before Hardy Cross, Hon. M. ASCE, revolutionized the analysis of statically indeterminate structures. The question arises as to whether the disadvantages of the classical methods might in this instance be overcome by the application of moment distribution. The answer is "yes." The independence of the torsional and bending moments in either of the beams of a right-angle double cantilever (mentioned by the author in the "Introduction") makes possible an "exact" solution in only one cycle of distribution and carry-over.

Mr. Cross had written[18] that

> "It is a simple matter to include the effect of torsion of connecting members in frame analyses provided we know the torsional properties of the members. Evidently the carry-over factor for torsion is unity, by statics."

One can see what Mr. Cross had in mind when one applies his method to the problem of this paper. In moment-distribution computations such as those that produced the results shown in Fig. 8(*b*), not only have the disadvantages of the classical method been overcome but all the advantages of moment distribution are gained. The procedure used by the writer involves the distribution of moments for no vertical deflection of C, and the distribution of arbitrary moments associated with a vertical movement at C, followed by a pro-

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[18] "Continuous Frames of Reinforced Concrete," by Hardy Cross and Newlin Dolbey Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932.

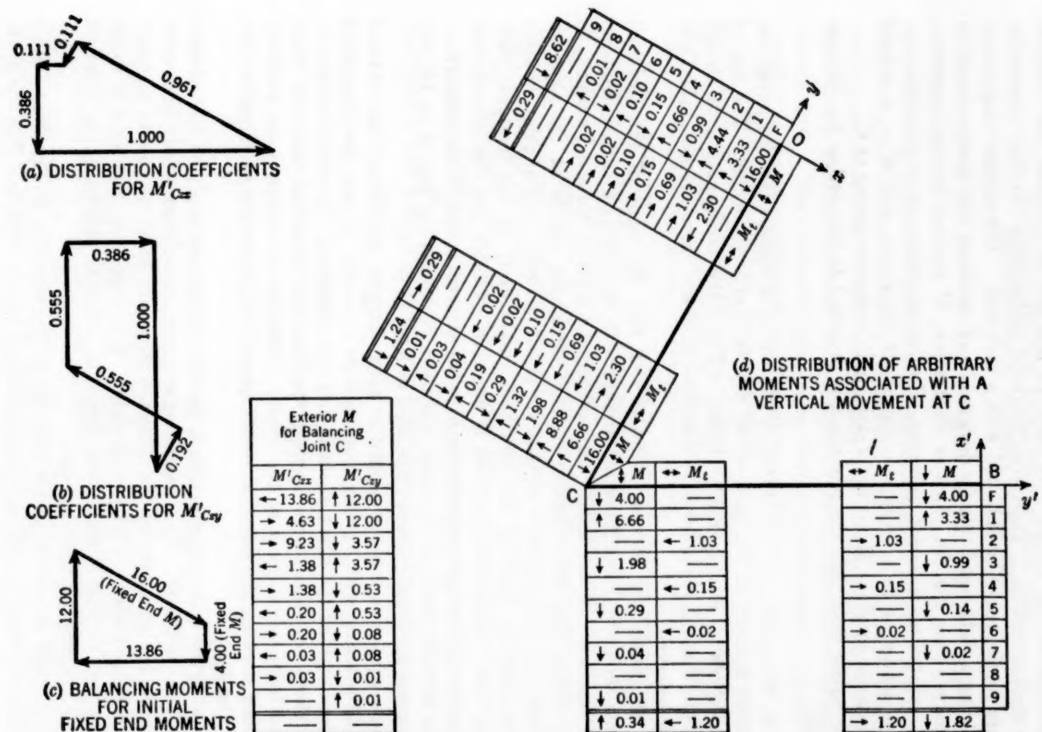

Fig. 9.—Moments, in Kip-Feet, Corresponding to the Reactions of Fig. 8(c)

portional correction of these moments which reduces the reaction at point C to zero.

Fig. 9 shows the computations that produced the results of Fig. 8(*c*). In this instance several cycles were required and the distribution was vectoral rather than numerical. There was, however, no question but that reasonably accurate results could be obtained with a slide rule. The author's suggestion of applying a right-hand rule for designation of moments was most valuable in this analysis (see Fig. 9). In this illustration, M denotes flexural moments, and M_t symbolizes torsional moments. In this particular set of computations, $M'_{C_{zx}}$ is defined as an exterior moment about the y'-axis, and $M'_{C_{zy}}$ is defined as an exterior moment about the x'-axis, both moments acting at C.

The writer feels that the profession is indebted to the author for his contribution.

F. E. Wolosewick,[19] M. ASCE.—The writer is pleased that his paper has stimulated interest in the analysis of double cantilever beams. The use of other methods for solving such problems, in lieu of the classical methods, should lead to more intensive studies of similar structures and more precise evaluations of their behavior.

As mentioned by Messrs. Conwell and Hu, the signs of the torsional moments shown in Table 2 are at variance with restraining torsional moments at the support. In the initial derivations, the internal torsional moment in the cantilever was used instead of the restraining couple at the support. This was done intentionally to indicate the direction of the angular rotation that the bending part of the structure would exert on the torsional part. The fact that the resulting answer is negative indicates that the internal torsional moment acts in an opposite sense to the external couple at the support.

The internal torsional moment was used in developing Eqs. 3 to 14, and the external torsional moment was used in Eqs. 15 to 17.

These equations were developed to solve complex cantilevers encountered in actual practice, with various combinations of loads. It was also considered important to take the effect of shear into consideration. In every instance in which these equations were applied, bending and torsional moments were checked from one support to the other. Deflected structures were similarly evaluated showing the torsional rotations and displacements produced by the applied loads.

Drawing the deflected structure in its entirety helps in applying the solution of a double cantilever. To a practicing engineer, no solution would be acceptable unless the geometry of the deflected structure could be clearly pictured and analytically checked for proper signs. It is especially important to know the variation of tensile and shear forces in concrete structures, in order to reinforce properly for these stresses.

Mr. Clough is correct in his analysis of the stability of Fig. 2(*g*). It is interesting to note, that for cantilevers of equal length, with load P at the apex, R_B is zero, R_A is equal to P, and the torsional moment at B is equal to P_a, and clockwise.

The writer wishes to express his thanks to the various commentators for the interest shown.

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[19] Structural Engr., Sargent & Lundy, Chicago, Ill.

DISCUSSION OF GROUP LOADINGS
APPLIED TO THE ANALYSIS OF FRAMES
PROCEEDINGS-SEPARATE NO. 183

BERNARD L. WEINER,[4] M. ASCE.—An excellent method of solving simultaneous linear equations occurring in the analysis of statically indeterminate structures has been presented by the author. Although equations of this type are not as important as they were before the development of the moment-distribution method, the need for a simple and direct method of solution will always exist. The analysis of structures having a large number of degrees of freedom of joint translation (sidesway) is a case in point.

[4] Cons. Engr., New York, N. Y.

Mr. Morrison's method, therefore, should be considered not as competing with moment distribution and similar methods, but as supplementing them. It is doubtful whether any method can compete with moment distribution in cases where the latter gives the complete solution.

It is unfortunate that Mr. Morrison did not use matrix algebra in his presentation. Although matrix algebra does not generally reduce the amount of numerical work required for a solution, it is an invaluable tool for study and research and often reveals useful properties which would otherwise remain hidden. This case is no exception in this respect and since the method lends itself readily to the use of matrices, the procedure would be clarified thereby.

Mr. Morrison states that the number of α_{ij}-values which can be chosen arbitrarily is $\frac{n(n+1)}{2}$. This is not completely accurate as can be seen by referring to Eq. 4 and expanding one typical equation to yield

$$X_a = \alpha_{aa} Y_a + \alpha_{ab} Y_b + \alpha_{ac} Y_c + \cdots \qquad (20)$$

Letting

$$Y_k = \frac{Z k}{\alpha_{aa}} \qquad (21a)$$

and substituting it into Eq. 20 results in

$$X_a = Z_a + \frac{\alpha_{ab}}{\alpha_{aa}} Z_b + \frac{\alpha_{ac}}{\alpha_{aa}} Z_c + \cdots \qquad (21b)$$

The value X_a will be fully determined if the α-ratios and Z_k are determined. Any other coefficient could obviously have been used instead of α_{aa} in Eq. 21a.

Actually, if one α-value in each equation is left undetermined, it will cancel out of the solution. Since any one α-value in each equation may be considered to be a factor, it can be given any value except zero, and the simplest value to assign to it is usually unity. There are therefore only $(n - 1)$-values of α in each equation to be determined and since there are n equations, it follows that there are a total of $n(n - 1)$ undetermined values of α_{ij}. Half of this number may be chosen arbitrarily, and the remaining half is determined from the conditions that γ_{ij} equals zero when i does not equal j, and that γ_{ij} does not equal zero when i equals j.

Matrix-algebra analysis reveals that, although the total number of α_{ij}-values which may usefully be chosen arbitrarily is $\frac{n(n-1)}{2}$, some choices are not permissible. A restriction on the choice of values is imposed because, in the solution, the α-coefficients separate into independent groups with n-elements in each group. Since one element must be made unity (or given some other convenient value) or left undetermined but cannot be set equal to zero, $(n - 1)$-elements remain to be determined in each group. It is obviously not permissible to use more than $(n - 1)$-conditions in one group and to compensate by using fewer conditions in another independent group. If this were done, contradictory results would be obtained in the former group, and all the x-values could not then be determined in the latter group.

Mr. Morrison is to be congratulated for presenting a basic concept which should lead to even more valuable results upon further study.

I. F. Morrison.[5]—Mr. Weiner's interest in the writer's work is appreciated.

[5] Prof. of Applied Mech., Dept. of Civ. and Muncipal Eng., Univ. of Alberta, Edmonton, Alta, Canada.

The method presented by the writer is not as "attractive" as it might be because the advantage of the method is not apparent when expressed in algebraic form. The use of matrix algebra was considered but was rejected as it was thought that there would be relatively few structural engineers who would be sufficiently familiar with this branch of mathematics. Moreover, the method of moment distribution is also not "attractive" when expressed in algebraic form.

In reference to the number of α-values which can be chosen arbitrarily, the writer fails to see how this number can be both $\frac{n\,(n+1)}{2}$ and $\frac{n\,(n-1)}{2}$. Obviously the number of values in the α-matrix is n^2, and it is also clear that none of the values on the main diagonal can be equal to zero. Examination of Table 2 reveals that all the values below the diagonal can be taken to be zero, and therefore the values above the diagonal must be computed. The number of these values is $\frac{n\,(n-1)}{2}$, and there are $\frac{n\,(n-1)}{2}$ equations from which they are obtained. Thus, the number of arbitrary values is

$$n + \frac{n\,(n-1)}{2} = \frac{n\,(n+1)}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

It has been suggested that other arrangements of the α-pattern may be more desirable because it is sometimes possible to obtain a more simple moment diagram which reduces the numerical work. An example of this kind of moment diagram is that for $Y_b = -1$ in Fig. 4(b). It is difficult to appreciate the process until one has worked directly with numerical values. When one is sufficiently familiar with the process, the α-values can be written by inspection from moment diagrams such as those given in Fig. 4.

Corrections for Transactions.—On page 6, in the paragraph immediately following Table 2, the expression $\frac{n\,(n+1)}{n}$ should be changed to $\frac{n\,(n+1)}{2}$. On page 7, directly below Fig. 2, the second sentence should be changed to "The length of the span is l, * * *." On page 9, the first sentence in the last paragraph should be changed to "The column under Y_b in Fig. 3(b) has been * * *."

DISCUSSION OF THE EQUIVALENT RECTANGLE IN PRESTRESSED CONCRETE DESIGN PROCEEDINGS-SEPARATE NO. 187

FRED E. KOEBEL,[2] J. M. ASCE.—Much valuable information has been con-

[2] Vice-Pres. and Chf. Engr., Prestressing, Inc., San Antonio, Tex.

tributed by the author to those engineers who are interested in the design of structural members made from prestressed concrete. Tabular and graphical solutions for sections have been sought by design engineers since the introduction of prestressing (on a large scale) into the United States (1946). The establishment of certain ratios (such as flange width to height, flange thickness to height, and web thickness to height) in Table 1, and the assumption of a ratio of width to depth for the equivalent rectangle (Fig. 1), limits the designer to a given section for any condition. In some instances, a design may require a specific headroom clearance; in other extreme instances, as much height is available as is desired by the engineer. This variation in requirements for the depth of the beam hinders the application of the assumed ratios. Unless tables and figures are available for all practical height variations, the design is likely to be uneconomical. Obviously tables and figures can be prepared for many variations. However, it is questionable if the preparation of such material is feasible. The practical use of the author's system would also be difficult in cases where composite action can be obtained—as in highway-girder design. Composite action is important and often leads to economical designs for highway-bridge spans.

In general, the live load or superimposed moment controls the required section modulus for post-tensioned girders, if provision is made for the loss of prestressing force required for the dead-load stresses and for the live-load stresses. This is true, because (generally) as the prestressing forces are applied, the beam rises from the soffit and the dead load stresses act simultaneously with those stresses caused by prestressing. Thus, the required section modulus can be approximated by use of

$$\frac{I}{c_b} = \frac{M_L}{K_1 f^b_c} \tag{17}$$

in which f^b_c is the allowable initial compression on the bottom fibers and K_1 is a factor based on the section properties and the percentage loss in stress from the initial prestressing force.

The designer can then obtain the required section modulus and can choose a cross-sectional shape. With the knowledge of the design limitations which are present, an approximation of the section can be made, and if the ratio of I/c obtained is of the same value as that shown or required, the stress conditions are confirmed. The design can then be refined and made practical for construction.

From Cols. 4 and 6, Table 2, it can be seen that for the beam having a 40-ft span, the section designation is D, and the depth h is 32.2 in. Therefore, from Table 1,

$$I = h^4\ (0.0392) = 42{,}500 \text{ in.}^4 \qquad (18a)$$

$$A = h^2\ (0.347) = 360 \text{ in.}^2 \qquad (18b)$$

$$c = h\ (0.622) = 20.1 \text{ in.} \qquad (18c)$$

and

$$r^2 = \frac{I}{A} = 118 \text{ in.}^2 \qquad (18d)$$

From Table 2,

$$M_L = 4.8 \times 10^6 \text{ in-lb} \qquad (19a)$$

$$M_D = 1.131 \times 10^6 \text{ in-lb} \qquad (19b)$$

and from Table 1,

$$W_D = h^2\ (0.362) = 378 \text{ lb per ft} \qquad (19c)$$

Therefore, the fiber stresses are as follows:

In the top fiber, under dead load,

$$f^t{}_D = 1.131 \times 10^6 \times \frac{12.2}{42{,}500} = +\ 325 \text{ lb per sq in.} \qquad (20a)$$

In the bottom fiber, under dead load,

$$f^b{}_D = 1.131 \times 10^6 \times \frac{20.1}{42{,}500} = -\ 536 \text{ lb per sq in.} \qquad (20b)$$

In the top fiber, under live load,

$$f^t{}_L = 4.8 \times 10^6 \times \frac{12.2}{42{,}500} = +\ 1{,}380 \text{ lb per sq in.} \qquad (20c)$$

In the bottom fiber, under live load,

$$f^b{}_L = 4.8 \times 10^6 \times \frac{20.1}{42{,}500} = -\ 2{,}270 \text{ lb per sq in.} \qquad (20d)$$

From the stresses given in Eqs. 20, it can be seen that even if no tensile stresses are allowed in the top fibers at initial prestressing, a greater eccentricity can be used than is shown by the author (5.79 in.). That is, the eccentricity can remain within the kern and a greater value can be obtained. The depth of the kern can be determined from

$$d_k = \frac{r^2}{y_t} = \frac{118}{12.2} = 9.7 \text{ in.} \qquad (21)$$

in which d_k is the depth of the kern and y_t is the distance from the CGG to the top fiber. If e is assumed to be 9.7 in., then the prestressing force required is

$$P = \frac{2{,}806 \times 360}{0.85 \left(1 + \dfrac{9.7 \times 20.1}{118}\right)} = 450 \text{ kips} \qquad (22)$$

Thus the stresses caused by only the prestressing are

$$f^t{}_p = \frac{450{,}000}{360}\left[1 - \frac{9.7\ (12.2)}{118}\right] = 0 \quad \text{(23a)}$$

and

$$f^b{}_p = \frac{450{,}000}{360}\left[1 + \frac{9.7\ (20.1)}{118}\right] = +\ 3{,}320 \text{ lb per sq in.} \quad \text{(23b)}$$

After losses the value of $f^b{}_p$ will decrease to 2,820 lb per sq in. Combining the stresses from the prestressing and the dead load results in

$$f^t{}_{Dp} = +\ 325 - 0 = +\ 325 \text{ lb per sq in.} \quad \text{(24a)}$$

and

$$f^b{}_{Dp} = -\ 526 + 3{,}320 = +\ 2{,}794 \text{ lb per sq in.} \quad \text{(24b)}$$

Combining the total-load stresses and the prestressing stresses after losses have occurred results in

$$f^t{}_{Tp} = +\ 1{,}705 - 0 = +\ 1{,}705 \text{ lb per sq in.} \quad \text{(25a)}$$

and

$$f^b{}_{Tp} = -\ 2{,}806 + 2{,}820 = +\ 14 \text{ lb per sq in.} \quad \text{(25b)}$$

However, even with an eccentricity of 9.7 in., the maximum possible eccentricity is not being used because the dead load comes on to the beam as the prestressing force is applied. Also, it is acceptable to allow some small amount of initial tension on the top fibers caused by the initial prestressing. Thus the eccentricity can be increased to 15 in. in which case the value of P would be 334 kips (including losses), $f^t{}_p = -\ 510$ lb per sq in. (after losses $f^t{}_p = -\ 434$ lb per sq in.), and $f^b{}_p = +\ 3{,}320$ lb per sq in. (after losses $f^b{}_p = +\ 2{,}820$ lb per sq in.). Combining stresses leads (before losses) to $f^t{}_{Dp} = -\ 185$ lb per sq in. and $f^b{}_{Dp} = +\ 2{,}784$ lb per sq in. After losses the stress $f^t{}_{Tp} = +\ 1{,}289$ lb per sq in. and $f^b{}_{Dp} = +\ 14$ lb per sq in.

Thus, by taking full advantage of the available section, the magnitude of the prestressing force is 34% less than that given by Mr. Peebles. This reduction will result in a large saving in the final cost of the member. The cables are, of course, laid in a catenary so that the stresses are within the allowable limit along the entire length of the beam. Even with pretensioning, the results from the investigation in which $e = 9.7$ in. indicate that a much larger value of e could be used. It would appear, therefore, that the use of the equivalent rectangle is not valid when applied to unsymmetrical sections. This invalidity is caused by the fact that the center of gravity of the rectangle is not the same as the center of gravity of the unsymmetrical section. Since these centers are not the same, the resulting stress conditions cannot be the same and the equations developed fail to give satisfactory results.

It should be realized that prestressed concrete is a very adaptable material; it should not be restricted to parameters for any given conditions. The skillful designer can make use of many practical shapes, the end result being a structurally sound and economical design. This takes no special skills, charts, or tables, but it does require a firm knowledge of the basic fundamentals of structural analysis.

JOHN J. PEEBLES,[3] J.M. ASCE.—Mr. Koebel has increased the value of the

[3] Engr., Bechtel Corp., San Francisco, Calif.

writer's paper by indicating that, although the design procedure proposed for the case in which the live load controls the design satisfies the stress limitations, a greater eccentricity and a smaller prestressing force could satisfy these limitations with a saving in steel and prestressing costs. However, the results that Mr. Koebel obtained are not entirely satisfactory, and the statements concerning these results are not correct (in some particulars) for the following reasons:

1. The results obtained by the proposed method are the final stress conditions, all prestress losses having taken place.
2. The stresses must be within the allowable limits, not only when the live load is on or off the member after prestress losses have occurred, but also when the member is subject only to the dead load—before prestress losses have taken place. Thus, by increasing the eccentricity to 15 in. (in the second example), the top fiber is stressed to −185 lb per sq in. and the bottom fiber is overstressed by 24% in compression under dead load before prestress losses take place; and the top fiber is stressed to −90 lb per sq in. under dead load even after prestress losses have taken place. Although Mr. Koebel states that it is acceptable to allow some tension in the top fibers of the member, it is doubtful whether this tension would be permitted by the average building code or design specification; tension is not permitted in the design of conventional reinforced-concrete structures.
3. The statement that the center of gravity of the equivalent rectangle is not the same as the center of gravity of the unsymmetrical section is in error. It is a geometrical fact that the centers of gravity coincide, as shown by Eqs. 6a and 7. From these equations the dimensions of the equivalent rectangle are obtained by making its area and moment of inertia equal to the area and moment of inertia of the given symmetrical or unsymmetrical section, both moments of inertia being about their respective centers of gravity. A reduction in the prestressing force was shown by Mr. Koebel, not because of a disparity between the centers of gravity of the sections, but as a result of increasing the eccentricity in the given section and reducing the prestressing force to obtain certain predetermined stress conditions. Also, the fact that the particular shape was unsymmetrical was of no consequence.

In presenting his material, the writer endeavored to illustrate the general application of the theory of the equivalent rectangle. Refinements in design—such as minor adjustments in cross sections and accounting for prestress losses—were purposely omitted (although important) to simplify the presentation. Rarely does any design method produce the final desired results without the need for some refinements and adjustments to comply with special conditions. It is agreed that no design method or short cut should take the place of sound engineering judgment.

Mr. Koebel suggests proportioning the cross section by trial and error to obtain the required section modulus. This may be a relatively simple operation for rectangular sections, but for I, T, hollow-rectangular, and similar irregular shapes, considerable time and energy may be expended before a satisfactory shape is obtained. One of the main advantages of the proposed method of design is virtually to eliminate such tedium.

It is not claimed that any one of the limited number of cross sections shown in Fig. 4 is the most economical—a term which is relative, anyway, and depends on the specific application. Probably "efficiency" would be a better term to characterize the cross sections in Fig. 4. In general, the cross sections increase in efficiency from A to G. There is no limit to the number of variations in shapes that could be devised. Contrary to Mr. Koebel's statement that sectional ratios limit the designer to a given section for any condition, there actually would be less limitation with a reasonably wide selection of shapes than there is with structural-steel shapes.

The T-shape (D in Fig. 4) can be used for floor systems and bridge decks so that advantage can be taken of composite action. In themselves, unsymmetrical sections of this type generally are less efficient than symmetrical sections. However, by incorporating the top flange in the slab or deck, efficiency usually can be attained.

When the live-load section modulus (computed from Eq. 12*b*) is greater than the total-load section modulus (computed by Eq. 11*a*, in which K reflects a reduction in section modulus as a result of the prestressing force), the required height of the section is controlled by the live-load fiber stress. This case will not be encountered often because it indicates that the ratio of live load to dead load is large, a condition usually associated with short spans. Although prestressed concrete members certainly are not limited to long spans, the greatest economy is generally demonstrated by using the members in this way.

In the usual case, in which the live load does not control the design, no adjustment in the cross section will be required to compensate for prestress losses resulting from both the plastic flow in the concrete and the creep in the steel; and the eccentricity, of course, already has its greatest attainable value at the point of maximum moment. Thus (in the first example), if it is desired to check the fiber stresses at the time of the initial prestressing, the required force P should be 626,000 lb. This force will result in top and bottom fiber stresses (when the beam is under the influence of both the prestress and dead load) of +790 lb per sq in. and +1,330 lb per sq in., respectively—each stress being less than the allowable stress of +1,800 lb per sq in.

To refine the design in the less common case in which the live load controls the design, it is usually necessary to reduce the allowable stress by from 15% to 20% in computing the live-load section modulus; otherwise the bottom fiber will be overstressed during initial prestressing, as was indicated in Mr. Koebel's computations. Thus, in the second example, by reducing the allowable concrete stress by 400 lb per sq in., the life-load section modulus would be 2,600 in^3. In a manner similar to that used in the second example, a slightly greater value of h equal to 34.5 in. is obtained. In order to establish the optimum value of e so that the top fiber stress will be zero and the bottom fiber stress will be f_c (under the influence of dead load, before prestress losses have occurred), the expression for the required K-value should be

$$K = \left(\frac{i}{1 - i}\right)\left(\frac{M_D}{f_c Z_b} + 1\right) \qquad (26)$$

in which Z_b is the section modulus for the bottom fiber. It should be noted that the only reason that the value of K must be changed from the value shown in Table 1 is that the value of i_2, which is a measure of e, has changed (Eq. 9c). All other factors remain the same as indicated in Table 1. The value of e from Eq. 15 is therefore 14.00 in. The required initial prestressing force is

$$P = (1 - i) f_r A = 350{,}000 \text{ lb} \qquad (27)$$

To compute stresses for this special case, Eq. 4 should be used instead of Eq. 16. Checking by use of Eq. 4, the stress in the top fiber is zero and the stress in the bottom fiber is +2,250 lb per sq in. under dead load and before prestress losses have occurred.

The prestressing force will reduce to 300,000 lb after losses of about 15%. From Eq. 4, the top fiber stress is +35 lb per sq in., and the bottom fiber stress is +1,850 lb per sq in. (under the influence of prestress and dead load). Under full load the top fiber stress is +1,170 lb per sq in., and the bottom fiber stress is zero.

Thus, the prestressing force has been reduced considerably, as advocated by Mr. Koebel, and also the limiting stresses of zero and +2,250 lb per sq in. have not been breeched under the influence of dead load before losses or under the influence of dead load or total load after losses.

Although there are those engineers who are apparently reluctant to utilize new design methods, charts, and tables to increase the efficiency of their operations, it is believed that the material presented by the writer can be used to simplify greatly the design of prestressed concrete members. As with most design methods, the results obtained may require minor adjustments to fit specific conditions. However, the bulk of the work required in the usual trial-and-error method of proportioning the cross section is eliminated by this design procedure, and the resulting stresses are within the allowable limits.

Corrections for Transactions.—On page 3 of the original paper references are made to I-sections. These should be changed to T-sections, and the caption of Fig. 2 should be changed similarly.

DISCUSSION OF WIND LOAD ON TRUSS BRIDGES
PROCEEDINGS SEPARATE NO. 201

F. B. FARQUHARSON, M. ASCE — It is encouraging to see new data becoming available regarding the wind forces to be expected on typical modern bridge sections, and the author is to be congratulated on this latest contribution. However, the writer would like to urge further investigation regarding the effect of Reynolds' number on structures of this type, where the shapes presented to the wind are of a blunt form.

The writer has made a considerable search of the literature, and finds only very meager test information on the effect of Reynolds' number (R) on flat plates, presented normal to the wind. Indeed, there seems to be only one test available in this region, which was reported by Flachsbart in 1925 [1]. He has stated that the drag coefficient, $C_D = 1.96$ at $R = 6{,}000$, and remained constant at this value down to $R = 4{,}000$. Below $R = 3{,}000$, C_D showed a marked drop, and at $R = 1{,}600$ it reached a value of 1.67. With further reduction in R, it was reported that C_D increased. These data are plotted in Figure 1. [2]

In connection with recent investigations on vortex frequency in the wake of wings, Krzywoblocki has stated that it is the belief of many investigators that in the case of bodies with sharp edges the vortex system is geometrically similar over the whole range of Reynolds' number, except at very low values. [3] This is equivalent to an assumption that the drag coefficient (C_D) for such sharp-edged bodies is constant at all Reynolds' numbers above a very low value. The investigations of Flachsbart and Krzywoblocki are based on the classical theory of von Kármán [4] which derived the mechanism of fluid resistance from a study of the vortex pattern in the wake of the body. Subsequent tests on circular cylinders revealed two critical regions on the curve plotting drag coefficient, C_D, against the log of Reynolds' number. In Figure 1, C_D for a circular cylinder is plotted against log R, and it is evident that a pronounced drop occurs in the curve in the region of $R \approx 1{,}800$, and that the curve falls off very sharply at $R \approx 200{,}000$. [5]

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1. Zeitschr. f. angew. Math. m. Mech., 15 (1925), pp. 32-37. Also see "Modern Developments in Fluid Mechanics," edited by S. Goldstein, Oxford University Press, 1938, Vol. 2, p. 423.
2. $R = 6{,}380\ VL$, where V is wind velocity in feet per second and L is the depth of the flat plate in feet.
3. M. Zbigniew Krzywoblocki, "Investigation of the Wing-Wake Frequency with Application of the Strouhal Number," Jour. Aer. Sci., Vol. 12 (Jan., 1945), p. 31.
4. Th. von Kármán, Nachr. Ges. Wiss. Gottingen (German), 1911, p. 509; 1912, p. 547; also Kármán and Rubach, "On the Mechanism of Fluid Resistance," Physic. 2, p. 49.
5. E. F. Relf, A. R. C., Reports and Memoranda (British) No. 102 (1914).

At a later date it was demonstrated that the Strouhal number, $S = fd/V$, is constant for a circular cylinder over a range of Reynolds' numbers of approximately 1,000 to 50,000, but increases sharply as C_D falls off at $R \approx 100{,}000$.[6] This information is also plotted in Figure 1. Clearly, for these critical values of Reynolds' number for a circular cylinder there is a drop in C_D, accompanied by a rise of the vortex frequency in the wake.

Relf and other experimenters in this field have revealed no evidence regarding the excitation of regularly shed vortices for a circular cylinder beyond Reynolds' numbers of approximately 100,000.

However, Pagon, while observing a smoke pattern around a pair of 11-ft. diameter steel stacks in 53-mph wind, found a distinct periodic vortex pattern at R = 5,500,000, where the vortex frequency had not increased as in Figure 1, but corresponds to a Strouhal number of 0.188.[7]

Glenn B. Woodruff has recently reported the observation of a well-developed vortex pattern behind an 11.4-ft. stack at two different wind velocities. At V = 37 fps and Strouhal number S = 0.225, R = 2,670,000, while at V = 51 fps and R = 3,750,000, S = 0.267. The Pagon and Woodruff data are plotted in Figure 1, and evidently fall well outside of the region reported by Relf and Simmons.

The reported information which produced the C_D curve for a circular cylinder in Figure 1 indicates clearly a critical region at a low Reynolds' number, and another, more marked, drop in C_D at a higher Reynolds' number. The wind data contributed by Pagon and Woodruff suggest that the upper critical region may be as high as R = 2,500,000. The meager data available for C_D on a flat plate show a drop at $R \approx 1{,}800$, corresponding to the drop in the C_D curve for a circular cylinder. <u>The dashed projection of this curve to higher values of R is pure conjecture.</u>

It would appear that the author's experiments were run at Reynolds' number between 30,000 and 80,000, depending on wind velocities and the depth of the principal vortex-producing surfaces normal to the wind. Thus these tests fall above the lower critical regions shown in Figure 1 for a flat plate, but below the upper critical value of C_D for a circular cylinder. The marked similarity between the known part of the flat plate curve and the corresponding portion of the curve for the circular cylinder leaves the unexplored upper end of the flat plate curve in a most unsatisfactory state. It would seem to be most important to have more test data on the drag coefficients for flat surfaces at very high Reynolds' number. If such tests were to substantiate the statement of Krzywoblocki previously noted, one would then turn with full confidence to the test data obtained at lower Reynolds' number.

In some regions it may be reasonable to design for wind velocities as high as 100 mph, and one might anticipate some truss members with depths as great as 3 to 4 feet. In modern girder bridges, the depths, of course, might exceed 15 feet. At three-foot depth at 100 mph, R = 6,380 x 146.5 x 3 = 2,800,000, while at 15 feet and 100 mph, R = 14,000,000. Such values of R are so far beyond any available test data or observation, that it would seem desirable to obtain data on C_D for sharp-edged sections up to at least R = 10,000,000.

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6. Relf and Simmons, A. R. C. Reports and Memoranda (British) No. 917 (1924).
7. W. W. Pagon, "Aerodynamics and the Civil Engineer," Eng. News-Record, July 12, 1934.

Such a series of tests would have presented a formidable problem several years ago, but in these days of large, high-speed wind tunnels, with balance systems designed for supersonic loads, the problem is comparatively simple. For example, a 30-inch, 121-pound steel beam tested at velocities between 50 and 450 mph would yield the values of R shown in Table I.

TABLE I

Reynolds' Number and Drag on 30" I Section

5 Ft. Long at Various Velocities

$R = 6{,}380\ VL$ $\quad L = 2.5'$ $\quad R = 15{,}950\,V$

V mph	V ft./sec.	R (dimensionless)	Drag* lb.
50	73	1,164,000	126
100	147	2,340,C00	512
200	293	4,670,000	2,034
300	440	7,020,000	4,570
400	587	9,360,000	8,150
450	660	10,500,000	10,320

*$D = C_D \ \rho/2 \ V^2 \ A$ where $C_D = 1.6$ and $\rho = 0.00237$ lb. ft.$^{-4}$ sec.^{2}

The fourth column in Table I indicates the approximate load the balance system would have to sustain for a beam 5 ft. long in order to develop the Reynolds' numbers listed in the second column. A test specimen as short as the one proposed here would require appropriate end plates to eliminate end losses.

This general question of the reliability of model tests was of great concern in the studies leading to the design of the New Tacoma Narrows Bridge. A series of static tests were run on the largest model which could be mounted in the available wind tunnel. However, the capacity of the balance system limited the top velocity to 153 mph, corresponding to a Reynolds' number of 59,700 based on the depth of the top chord of the truss.[8]

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8. F. B. Farquharson, "Aerodynamic Stability of Suspension Bridges," Bull. 116, Part IV. Eng. Exp. Sta., University of Washington. (In Press)

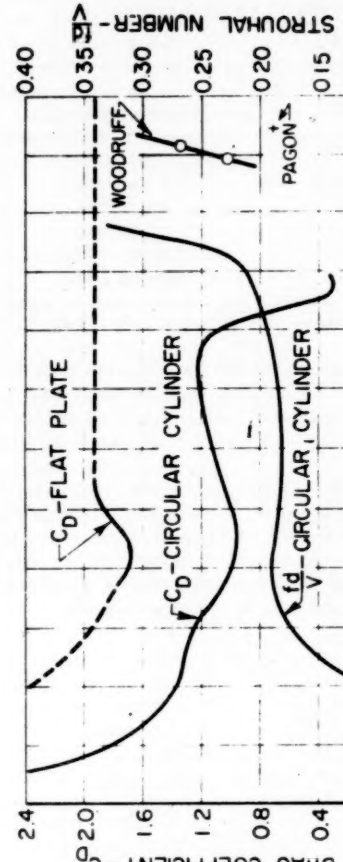
DRAG COEFFICIENT - C_D
STROUHAL NUMBER - $\frac{fd}{V}$
C_D - FLAT PLATE
C_D - CIRCULAR CYLINDER
$\frac{fd}{V}$ - CIRCULAR CYLINDER
WOODRUFF
PAGON
LOG_{10} R
2.4
2.0
1.6
1.2
0.8
0.4
0
0.40
0.35
0.30
0.25
0.20
0.15
0.10
1
2
3
4
5
6
7

Figure I

DISCUSSION OF INVESTIGATION OF THE DEAD LOAD STRESSES IN THE MISSISSIPPI RIVER VETERANS MEMORIAL BRIDGE AT EAST ST. LOUIS, ILLINOIS PROCEEDINGS SEPARATE NO. 219

ALMON H. FULLER, M. ASCE — The six conclusions suggest a careful analysis of the results of a well prepared and ably carried out investigation of this interesting bridge. They command confidence. The recognized error of 1000 psi is in accord with other field investigations of strains and stresses.

In the "Synopsis" are given two figures of interest which are not included in the "Conclusions." The 3000 psi for temperature and erection effects is simply a different wording of conclusion five. The statement that the measured stresses were lower than the computed dead load stresses by about 1000 psi is apparently given as a statement of fact without discussion. To the writer that point is of great interest. Some of the few available results of field investigations show differences between observed and computed stresses of various magnitudes but mostly in the same direction. Among the reasons for such differences are: 1. The retarding effect of deflection because of the bending of the members (secondary stresses), and 2. The fact that the deformation of a steel member under stress is dependent (even if to an unmeasured extent) upon the time under stress. The physicists maintain that steel is not 100% elastic and, therefore, that deformation is dependent upon time.

There is reason to believe that the retarding effect of the bending of members in a trussed structure is within 5% of the total deflection and, therefore, reduces the direct stresses by about the same amount. The reduced observed stresses, reported as about 1000 psi, are somewhere near 10% of the highest stresses which are reported. Therefore, in the few members under consideration more than 5% of the difference between observed and computed stresses remain unaccounted for.

The writer, in 1925, conducted some field observations on strains and stresses in the Equitable Building in Des Moines, Iowa during erection. Observed stresses exceeded the computed ones by much more than 10% [1,2]. C. T. Morris in 1928 conducted similar observations during the erection of the American Insurance Union Building in Columbus, Ohio, and found about the same comparative results. In his report [3] he included a discussion of our results with his. He put the relation between computed stress and observed strain in the form of a "modulus of elasticity" as representing the relation of stress and strain over a number of weeks in large built columns. The "modulus" as reported by Morris varied from below 20,000,000 psi to over 24,000,000 psi with average unit stresses varying from about 6000 psi to 9000 psi. The duration of time is not recorded but runs over a season of a number of weeks.

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1. Bulletin 72, Iowa Engineering Experiment Station, Iowa State College, 1925.
2. Engineering News Record. Vol. 93 (1924) p. 540, 655.
3. Bulletin 40, Engineering Experiment Station, Ohio State University, 1928.

The writer does not find in this report or in some other reports of field investigations, any indication of continued deformation under stress. Nevertheless he believes it exists even if so small as to be accepted as a part of inevitable lack of precision as in the paper under discussion. This is one phase of many where current methods of computation charge the structure with greater stresses than it will be required to carry. This point might well be kept in mind when it becomes economical to do more refined designing than is required by present specifications.